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The spectral broadening of sound by turbulent shear layers. Part 1. The transmission of sound through turbulent shear layers

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The transmission of sound through a turbulent shear layer is reviewed both with respect to the mathematical theory of the distortion of signals during propagation and also as the basis of the analysis in part 2 of the spectral broadening evinced by experimental results relevant to the study of aircraft noise. The reflexion and transmission coefficients, which involve both amplitude and phase changes, are obtained for scattering by an irregular and unsteady interface convected between two media; diffraction of high frequency sound by small-scale turbulence in a shear layer is accounted for by means of a phase shift and conservation laws for energy, wavenumber and amplitude. These results may be used to construct the sound field transmitted through a turbulent shear layer from a source in the interior of a jet; multiple internal reflexions are accounted for in the case of transmission through two parallel shear layers, i.e. a jet of finite width shielding a noise source. A statistical description is given of the process of attenuation of the transmitted wave as energy is diffracted by the turbulence, and of the partial compensation by interference between correlated components of the refracted wave; the reference case is a monochromatic point source, which, when placed behind a system of shear layers, has its energy redistributed directionally and spread over a range of frequencies. The expressions obtained for the energy flux include as particular cases the results of Howe for the plane (1975) and impedance (1976) layers, the effects of turbulence being shown to be consistent with the experiments of Schmidt & Tilmann (1970) and Ho & Kovasznay (1976*a*, *b*).

1. Introduction

The purpose of this paper is to review certain aspects of the mathematical theory of the propagation of waves (signals with information content) through a non-uniform medium. Gradual or sharp changes in the properties of the latter, either dynamical or constitutive, distort the original signal and cause coherent energy to be transformed into 'noise'. We shall be concerned specifically with the redistribution in direction and frequency of sound transmitted through turbulent shear layers.

1.1. *Scattering and diffraction of waves*

The scattering of waves by various bodies with regular shapes has been studied since the work of D'Alembert (1747) by solving the wave equation, together with the appropriate boundary conditions, in suitable co-ordinate systems. The radiation con-

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dition was introduced by Sommerfeld (1896), whose results for diffraction by a semi-infinite sheet have led to a few extensions, e.g. to the thick plate (Jones 1953), by various methods (Crighton & Leppington 1973). Gradual changes in the properties of the wave-bearing medium are described by space or time variations in the coefficients of the appropriate wave equation and some exact solutions have been obtained for propagation in a stratified medium (Brekovskikh 1960). The simplest problem is a plane interface, which gives rise to reflected and transmitted plane waves for each incident plane-wave component, e.g. of a spherical wave front forming a lateral wave (Landau & Lifshitz 1959, § 72).

In general, physical problems involving arbitrary variations in the medium have been analysed successfully only when the wavelength is *not* of the same order as the length scale of the inhomogeneities. This corresponds to two limiting cases: at low frequencies the properties of the medium change abruptly on the scale of a wavelength, and non-uniformities appear as an interface, generally of irregular or unsteady shape, scattering the waves; at high frequencies, e.g. in the diffraction of sound by non-uniform flow (Blokhintsev 1956), the geometrical approximation of propagation along rays can be applied. In certain refraction problems scattering and diffraction are both involved, a case of practical interest being a shear layer, which may be modelled as an irregular unsteady interface entraining a region of anisotropic and possibly non-stationary, turbulence.

In practical situations the non-uniformities occur randomly; for example, irregularities of an ocean surface return clutter with a sonar echo (Clarke 1973), atmospheric turbulence produces fluctuations in the reception of radio signals (Tatarski 1967) and impurities in the glass limit the resolving power of lenses (Chernov 1967). The scattering of sound has been considered not only for non-uniformities of composition but also with regard to turbulence and shock waves (Lighthill 1953), and a kinetic-theory method may be used (Howe 1973*a*, *b*). The last reference considers multiple scattering, an example of which is the reflexions that can occur in the interior of a double-sided region, such as a slab of material acting as a noise shield (Howe 1976). In the model of a plane moving interface (Miles 1958; Howe 1975), there is the implied question of the possible effect of sound on the instabilities of a tangential discontinuity of velocity (Jones & Morgan 1972), which does not appear to have been examined with regard to the randomly unsteady and non-uniform velocity profiles of a turbulent shear layer.

1.2. *Interpretation of 'noise' as information*

The noise developed by a signal may be interpreted as furnishing information regarding those properties of the wave-bearing medium that are responsible for refraction, i.e. scattering and/or diffraction. This point of view is illustrated by the case of sound propagating across a turbulent shear layer which forms the mixing region of a jet. If the ambient medium has different physical properties, then at low frequencies the changes in density (and wave speed) may be assumed to occur fairly abruptly across an interface of irregular and unsteady shape. The entrainment of the ambient fluid by the interface may be represented by a region of turbulence, which introduces an additional refraction mechanism in the form of localized velocity fluctuations small compared with the mean convection velocity but with random directions.

A wave component incident on a turbulent shear layer will emerge with a changed direction of propagation and a distinct frequency determined by the time-dependent diffractive crinkling of rays by turbulence and the location on the interface at which scattering occurs. These effects are described by amplitude and phase changes, applying differently to each wave component of a coherent beam emitted, say, by a source in the jet and received in the ambient medium as an incoherent bundle. If the source and observer are both sited in free space, but separated by a jet acting as a shield, the transmission through *two* turbulent shear layers will result in increased incoherence. In the case of double or multiple shear layers it is necessary to ascertain whether multiple internal reflexions can modify significantly the level of noise generated, viz. because they may involve a significant fraction of the energy which is ultimately transmitted.

The particular random realization of the turbulent shear layer determines the wave field, but the mean radiated energy, which is quadratic, depends on only the correlations within the turbulence and the interface. This leads one to a statistical description of both the attenuation of a coherent incident beam and the interference between correlated components of the transmitted bundle, which tends to preserve *some* of the energy otherwise absorbed or backscattered by the turbulence. These effects appear in the formal analytical expression for the radiation received by an observer in the ambient medium, and may be illustrated by an example involving an isotropic monochromatic wave emitted by a point source within the jet and observed in the ambient medium, where its energy is distributed anisotropically in space and spread over a spectrum of frequencies.

1.3. *Signals in nature and for tests*

The noise associated with an initially coherent signal as a result of propagation or interference is usually regarded as an undesirable degradation of the quality of information being collected. However, if the physical process of distortion is studied the noise may be estimated and eliminated from the signal received in order to recover as much as possible of the original information. Furthermore, the noise, being itself a function of the properties of the medium, provides information about the conditions between the source and the observer. As an example we cite the problem of estimating the energy of the original signal from a star by taking account of the noise accreted during propagation in the galaxy or through intergalactic clouds, the latter possibly giving some indication of the velocity of scattering particles in outer space.

This approach is used in terrestrial conditions; viz. by transmitting a known test signal through a substance to ascertain its properties without the necessity of taking measurements in its interior. In laboratory conditions an appropriate choice of signal can give high accuracies, e.g. the acoustic determination of the gas constant (Chandler, Colclough & Quinn 1976). In industrial conditions the temperature of hot materials or the composition of easily contaminated chemicals may be similarly determined without recourse to probes. In geophysical studies of atmospheric events or of the structure of the earth's crust a single signal can provide an indication of the average conditions along its path of propagation.

This introduction (§ 1) suggests that the following sequence of topics be discussed: the determination of the effective reflexion and transmission coefficients which describe scattering and diffraction of a plane wave component (§ 2); the construction on this basis of the fields refracted by single or double turbulent shear layers (§ 3); the application of the statistical properties of the medium to determine the directivity and spectrum of the transmitted radiation (§ 4). A preliminary verification of the theory with reference to experiment is also included (figure 3). The analytical methods would apply generally to acoustic, elastic and electromagnetic waves in, respectively, fluids, solids and dielectrics with analogous random properties, but bearing in mind the subsequent application (in part 2) to the spectral broadening of experimental and aircraft noise, we shall concern ourselves with the equations and the terminology of aerodynamic acoustics.

2. The reflexion and transmission coefficients

For the purpose of examining the scattering of sound, our theoretical model of the shear layer will be assumed to consist of an irregular and unsteady interface across which the properties of the flow change discontinuously. This model suffices when the wavelength greatly exceeds the mean shear-layer width, but at higher frequencies it is necessary to take account of the details of the flow in the interior of the shear layer. The correction for the propagation of higher frequency sound through the turbulent shear flow will be based on the approximation of geometric acoustics.

2.1. *Scattering by a moving interface*

We consider (figure 1*a*) an interface of irregular and unsteady shape whose mean position is the plane $x_3 = 0$ and whose height, i.e. the vertical displacement from $x_3 = 0$, is given by $x_3 = \xi(\mathbf{y}, t)$. If the interface lies between a stationary ambient medium and a jet of velocity $\mathbf{V}$, we assume that it convects with velocity $\alpha\mathbf{V}$ $(0 < \alpha < 1)$ and that its height depends on the moving co-ordinates $\mathbf{y} = \mathbf{x} - \alpha\mathbf{V}t$.† If the jet and ambient fluids are different we denote their mass densities and speeds of sound by (ρ, c) and (ρ_0, c_0) respectively. Of the dynamical variables, the pressure P and the normal displacement $Z_n = \mathbf{Z}.\mathbf{N}$ of a fluid particle are continuous at the interface, where $\mathbf{N}$ denotes the unit normal. These are related by the momentum equation

$$\rho(\partial/\partial t + \mathbf{V}.\nabla)^2\mathbf{Z} + \nabla P = 0, \tag{1a}$$

which for an incident plane monochromatic disturbance of radian frequency ω and wave vector $\mathbf{k}$ reduces to

$$Z_n = \{i(\mathbf{k}.\mathbf{N})P/\rho(\omega - \mathbf{k}.\mathbf{V})^2\}. \tag{1b}$$

The corresponding locally scattered waves are also plane if the interface is locally flat, i.e. its radius of curvature is much larger than the wavelength (Kirchhoff's scattering approximation, see Born & Wolf 1970, p. 378). Denoting by $\mathbf{k}_i$, $\mathbf{k}_r$ and $\mathbf{k}_t$ the wave vectors of the incident, reflected and transmitted waves respectively at the element of the interface with position vector $\mathbf{X} = (\mathbf{y}, \xi)$, continuity of pressure implies that

$$\exp(i\mathbf{k}_j.\mathbf{X}) + R\exp(i\mathbf{k}_r.\mathbf{X}) = T\exp(i\mathbf{k}_t.\mathbf{X}), \tag{2a}$$

in which the amplitude of the incident wave is taken to be unity and R and T denote respectively the local reflexion and transmission coefficients. The continuity of fluid-particle displacement along the normal requires that

$$\frac{\mathbf{N}}{\rho}.\left\{\mathbf{k}_i\frac{\exp(i\mathbf{k}_i.\mathbf{X})}{(\omega - \mathbf{k}_i.\mathbf{V})^2} + R\mathbf{k}_r\frac{\exp(i\mathbf{k}_r.\mathbf{X})}{(\omega - \mathbf{k}_r.\mathbf{V})^2}\right\} = T\frac{\mathbf{k}_t.\mathbf{N}}{\rho_0\,\omega^2}\exp(i\mathbf{k}_t.\mathbf{X}). \tag{2b}$$

† $\mathbf{y}$ is used in this sense throughout the analysis.

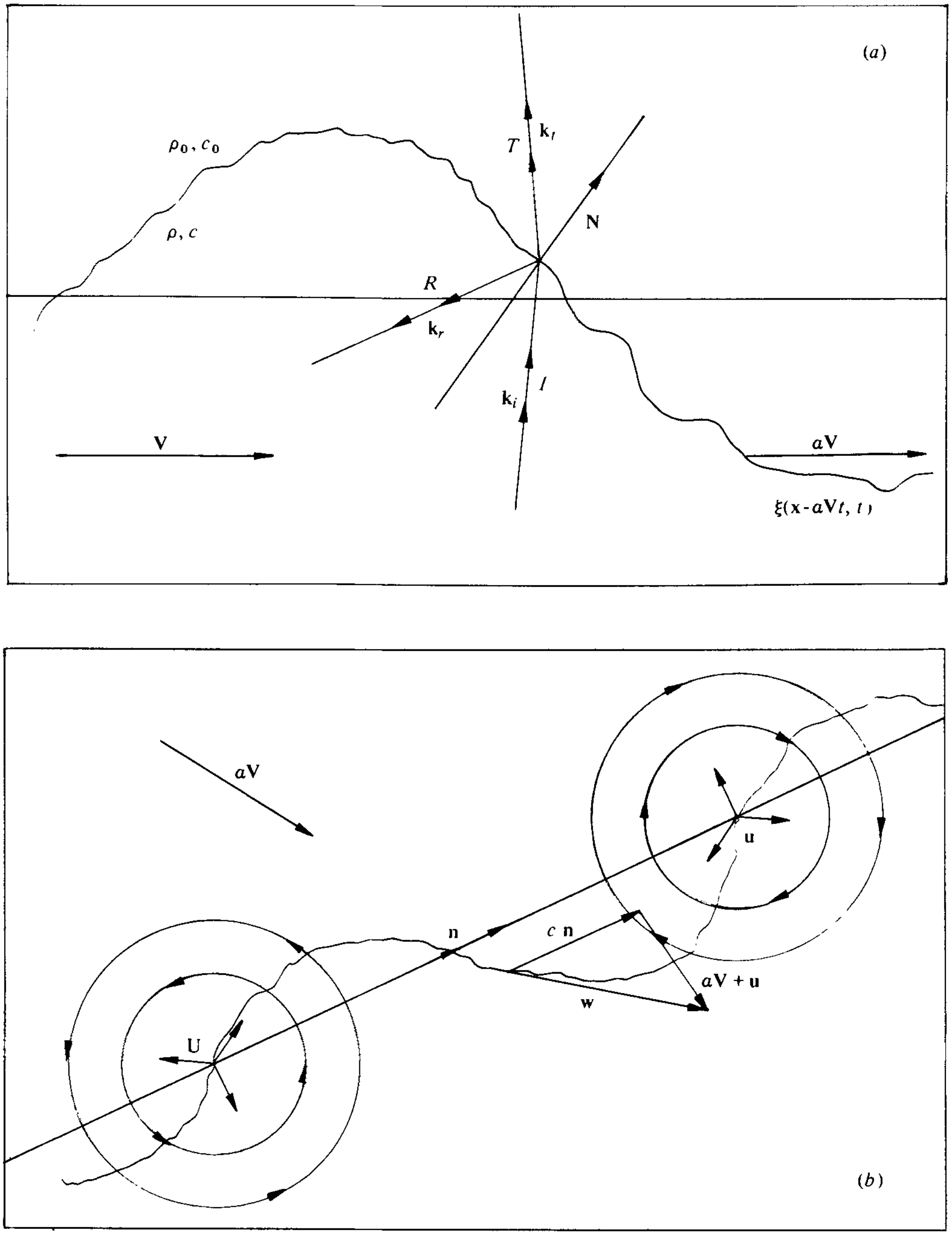


FIGURE 1. Refraction of sound by a shear layer. (*a*) Scattering by an irregular unsteady interface. (*b*) Diffraction of rays in a region of turbulence.

We assume in the first instance that the scattering elements may be regarded as horizontal (parallel to $x_3 = 0$), either because the undulations of the interface are shallow or because the irregularities, being random, have zero slope on average, the slope being correlated over scales much smaller than those for the interface height. Then the horizontal wave vector $\mathbf{g}$ (i.e. wave-vector components g_1, g_2) is conserved, whereas the vertical wavenumber (i.e. the component along the x_3 axis) is given by

$$\gamma(\mathbf{g}) = \{(\omega - \mathbf{g}.\mathbf{V})^2/c^2 - g^2\}^{\frac{1}{2}}, \quad \Gamma(\mathbf{g}) = \{\omega^2/c_0^2 - g^2\}^{\frac{1}{2}}, \tag{3a, b}$$

respectively, for the incident and transmitted wave and by $-\gamma$ for the reflected wave. Thus setting $\mathbf{k}_i = (\mathbf{g}, \gamma)$, $\mathbf{k}_r = (\mathbf{g}, -\gamma)$ and $\mathbf{k}_t = (\mathbf{g}, \Gamma)$ when solving (2a, b) for the reflexion and transmission coefficients, we find

$$R(\mathbf{g}) = \frac{\rho_0\gamma - \rho(1 - M\cos\theta)^2\Gamma}{\rho_0\gamma + \rho(1 - M\cos\theta)^2\Gamma}\exp(i2\gamma\xi), \tag{4a}$$

$$T(\mathbf{g}) = \frac{2\rho_0\gamma}{\rho_0\gamma + \rho(1 - M\cos\theta)^2\Gamma}\exp\{i(\gamma - \Gamma)\xi\}, \tag{4b}$$

in which $\mathbf{M} \equiv \mathbf{V}/c_0$ is the Mach number of the jet and θ the angle between the velocity of the jet and the direction of incidence. The phases $2\gamma\xi$ for reflexion and $(\gamma - \Gamma)\xi$ for transmission (Howe 1976) are both determined by the product of the displacement of the interface and the respective differences in the vertical wavenumbers. In the present approximation the variation of the amplitude is specified by the properties of the media alone, and may be represented in terms of an amplitude factor which is equal to

$a_0(\mathbf{g}) = \rho\Gamma/\rho_0\gamma$ for media at relative rest (Rayleigh 1945, p. 80),

$a_1(\mathbf{g}) = (1 - M\cos\theta)^2 a_0(\mathbf{g})$ for incidence from within the jet (Howe 1975)

and $a_2(\mathbf{g}) = (1 - M\cos\theta)^{-2} a_0(\mathbf{g})$ for incidence from outside the jet.

In the three cases the reflexion and transmission factors, given respectively by $|R| = (1-a)/(1+a)$ and $|T| = 2/(1+a)$, imply that $1 + |R| = |T|$, and therefore that the energy is locally conserved during scattering by smooth or irregular interfaces.

The preceding model of an irregular interface as a planar array of horizontal scattering elements (Clarke 1973) can be refined within the confines of Kirchhoff's approximation to account for the slope as well as the height of each facet, as in a directed array. The wave vector is now conserved transverse to a sloping flat element, i.e. the difference between the incident $(\mathbf{g}_i, \gamma)$ and transmitted $(\mathbf{g}_t, \Gamma)$ wave vectors (which we have separated into horizontal and vertical components) lies along the normal to $x_3 = \xi$, which to $O(|\nabla\xi|^2)$ is specified by $\mathbf{N} = (-\nabla\xi, 1)$, i.e.

$$(g_{t_1} - g_{i_1})(\partial\xi/\partial x_1)^{-1} = \gamma(\mathbf{g}_i) - \Gamma(\mathbf{g}_t) = (g_{t_2} - g_{i_2})(\partial\xi/\partial x_2)^{-1}. \tag{5}$$

We emphasize that the horizontal wave vector is *not* conserved: $\mathbf{g}_i \neq \mathbf{g}_t$ in (5). This, together with (3a, b), may be used to write the incident vertical wavenumber in the two forms

$$\gamma(\mathbf{g}_i) - \gamma(\mathbf{g}_t) = \{1 - \Gamma(\mathbf{g}_t)/\gamma(\mathbf{g}_t)\}\mathbf{K}.\nabla\xi + O(|\nabla\xi|^2), \tag{6a}$$

$$\mathbf{K} \equiv \mathbf{g}_t + k_t(c_0/c)\mathbf{M}, \quad k_t \equiv \{g_t^{-2} + [\Gamma(\mathbf{g}_t)]^2\}^{\frac{1}{2}}. \tag{6b, c}$$

The coefficient $\{\partial\gamma/\partial \mathbf{g}_i\}_{\mathbf{g}_i=\mathbf{g}_t} = -\mathbf{K}/2\gamma(\mathbf{g}_t)$ introduces the deviation vector $\mathbf{K}$, which varies between $k_t(c_0/c)\mathbf{M}$ (where $\mathbf{M} \equiv \mathbf{V}/c_0$ is the Mach vector of the jet) for vertical transmission to $k_t\{(c_0/c)\mathbf{M} \pm \mathbf{n}_t\}$ for propagation in grazing directions ($\mathbf{n}_t$ being the wave normal projected on the horizontal direction: $\mathbf{n}_t \equiv \mathbf{g}_t/g_t$). Equations $(2a, b)$ remain valid when $\mathbf{k}_i = (\mathbf{g}_i, \gamma)$ and $\mathbf{k}_t = (\mathbf{g}_t, \Gamma)$, and may be expressed in terms of $\mathbf{g}_t$ alone by use of (5) and $(6a)$, e.g. in the case of the phase shift Φ_a due to transmission across a sloping interface we have

$$\Phi_a(\mathbf{g}_t, \nabla\xi) = \{\gamma(\mathbf{g}_t) - \Gamma(\mathbf{g}_t)\}\{\xi - \mathbf{y}\,.\,\nabla\xi + \{\mathbf{K}/2\gamma(\mathbf{g}_t)\}\,.\,\nabla\xi^2\}, \tag{7}$$

where γ, Γ and $\mathbf{K}$ are specified by $(3a, b)$ and $(6b, c)$. The phase shift Φ_a is proportional, through the difference in the vertical wavenumbers (calculated from the horizontal transmitted wave vector $\mathbf{g}_t$), to the height ξ of the interface minus the projection $\mathbf{y}.\nabla\xi$ of the slope on the mean plane plus a cross-product $\nabla(\frac{1}{2}\xi^2) = \xi\nabla\xi$ involving the deviation vector $\mathbf{K}$.

2.2. *Phase diffraction in turbulence*

We now consider the correction to be applied to the above results in the case of short wavelengths, where the propagation of sound is affected by the turbulence within the shear layer (figure 1*b*). The local turbulent perturbation velocity relative to the velocity $\alpha\mathbf{V}$ of the mean shear flow is denoted by $\mathbf{u}$, and is of random direction $\mathbf{m}$ and magnitude $|\mathbf{u}| \sim \beta V$ $(0 < \beta < \alpha)$. If the Mach number of the turbulence $M' = |\mathbf{u}|/c$ is small the local flow may be regarded as incompressible. The propagation of high frequency sound in a non-uniform but homogeneous flow is described by the convected wave equation

$$\{[\partial/\partial t + (\alpha\mathbf{V} + \mathbf{u})\,.\,\nabla]^2 - c^2\nabla^2\}\Psi \exp(i\Phi) = 0, \tag{8}$$

for which a solution is sought in the form of a transmission coefficient consisting of amplitude and phase functions, denoted by Ψ and Φ respectively. On double application of a linear differential operator L, the real and imaginary parts of

$$\exp(-i\Phi)\,L^2\{\Psi \exp(i\Phi)\}$$

are, respectively, $L^2\Psi - \Psi(L\Phi)^2$ and $\Psi L^2\Phi + 2(L\Phi)(L\Psi)$. Noting that the wave equation corresponds to the operator equality $L_{\mathbf{x}}^2 = L_t^2$, where $L_t \equiv D/Dt$ is the material derivative and $L_{\mathbf{x}} \equiv c\nabla$, and equating the real and imaginary parts, we obtain

$$(D\Phi/Dt)^2 - c^2(\nabla\Phi)^2 = \Psi^{-1}\{D^2\Psi/Dt^2 - c^2\nabla^2\Psi\}, \tag{9a}$$

$$D^2\Phi/Dt^2 - c^2\nabla^2\Phi = -2\Psi^{-1}\{D\Psi/Dt\, D\Phi/Dt - c^2\nabla\Psi\,.\,\nabla\Phi\}, \tag{9b}$$

which describes the evolution of the amplitude Ψ and phase Φ. The perturbation of the mean flow is specified by the turbulent velocity $\mathbf{u}(\mathbf{y}, t)$ appearing in the material derivative $D/Dt = \partial/\partial t + \alpha\mathbf{V}\,.\,\nabla + \mathbf{u}\,.\,\nabla$.

The amplitude Ψ varies significantly on the scale of the flow l while the phase Φ changes by 2π over a wavelength λ, thus the left- and right-hand sides of $(9a)$ are respectively of orders λ^{-2} and l^{-2}. Since the wavelength is assumed to be small on the flow scale $(\lambda^2 \ll l^2)$ it follows that the phase satisfies the convected eikonal equation

$$\{\partial\Phi/\partial t + (\alpha\mathbf{V} + \mathbf{u})\,.\,\nabla\Phi\}^2 - c^2(\nabla\Phi)^2 = O(\lambda^2/l^2). \tag{10}$$

The derivatives of the phase with regard to position and time define respectively the wave vector $\mathbf{k}$ and minus the frequency $-\omega$, thus the compatibility relation

$$\partial^2\Phi/\partial\mathbf{x}\partial t = \partial^2\Phi/\partial t\partial\mathbf{x}$$

appears (Lighthill 1964) as conservation of the number of waves, with

$$\mathbf{k} \equiv \nabla\Phi, \quad \partial\mathbf{k}/\partial t + \nabla\omega = 0, \quad \omega \equiv -\partial\Phi/\partial t. \qquad (11\,a\text{–}c)$$

The geometrical-acoustics description of the propagation of sound along rays implies the conditions $(\nabla.\mathbf{k})^2 \ll k^4$, $(D\omega/Dt)^2 \ll \omega^4$ and $(D\mathbf{k}/Dt)^2$, $(\nabla\omega)^2 \ll k^2\omega^2$, which require that the frequency and the wave-normal direction $\mathbf{n} \equiv \mathbf{k}/k$ vary slowly on a scale of the wavelength or wave period. Substitution of (11a, c) in (10) leads respectively to the dispersion relation and the group velocity:

$$\omega(\mathbf{k}) = ck + (\alpha\mathbf{V} + \mathbf{u}).\mathbf{k}, \quad \mathbf{w} \equiv \partial\omega/\partial\mathbf{k} = c\mathbf{n} + \alpha\mathbf{V} + \mathbf{u}. \qquad (12\,a, b)$$

These equations show that rays issuing radially from a point source have a drift velocity $\mathbf{w}_* \equiv \mathbf{w} - c\mathbf{n}$ which is associated with transport by the shear-layer flow and consists of (i) uniform convection at the mean velocity $\alpha\mathbf{V}$, which preserves the straightness of the rays relative to the source, and (ii) random deviations by the turbulence $\mathbf{u}$, whose component $\mathbf{u}_s = \mathbf{u} \times \mathbf{n}$ transverse to the mean direction of propagation causes 'crinkling' of the ray paths.

In the presence of convection by the mean flow alone the eikonal equation (10) implies that $\mathbf{k}$ and ω remain fixed for propagation along the characteristics of (10), viz. the rays whose directions are defined by the group velocity (12b) with $\mathbf{u} = 0$. The inclusion of the turbulence introduces a disturbance $\Phi = \Phi_0 + \Phi_b$ and the exact eikonal equation (10) may be written as $\{\partial/\partial t + (\alpha\mathbf{V} + \mathbf{u}).\nabla\}(\Phi_0 + \Phi_b) = -ck$, in which to first order the total wavenumber $k = |\nabla(\Phi_0 + \Phi_b)|$ is approximated by

$$k = k_0 + \mathbf{n}_0.\nabla\Phi_b,$$

where k_0 and $\mathbf{n}_0$ are unperturbed quantities. Neglecting nonlinear terms and subtracting the equation for the mean state, we obtain the following equation for the phase perturbation:

$$\{\partial/\partial t + (\alpha\mathbf{V} + c\mathbf{n}).\nabla\}\Phi_b = \mathbf{k}.\mathbf{u} + O(|\nabla\Phi_b|^2), \qquad (13)$$

where $\mathbf{k}$ and $\mathbf{n}$ now denote respectively the unperturbed wave vector and the wave normal. Equation (13) states that, in a first approximation, the perturbation of the phase along a convected ray is caused by turbulent transport of the mean wave vector, corresponding to a retardation if $\mathbf{k}.\mathbf{u} < 0$ and an advance if $\mathbf{k}.\mathbf{u} > 0$. Introducing the derivative along the mean convected ray $d/dt \equiv \partial/\partial t + \mathbf{w}_0.\nabla$, with $\mathbf{w}_0 = c\mathbf{n} + \alpha\mathbf{V}$, and noting that $d/dt \simeq c\,d/ds$, where ds is the arc length, (13) is written in the form $d\Phi/ds = \mathbf{k}.\mathbf{u}/c$, and may be integrated along the ray to yield

$$\Phi_b(\mathbf{x}, t) = \beta M(c_0/c)\,k\zeta, \quad \zeta(\mathbf{x} - \alpha\mathbf{V}t, t) \equiv \int_{\mathscr{S}} \mathbf{n}.\mathbf{m}\,ds. \qquad (14\,a, b)$$

In (14a), Φ_b denotes the total phase shift caused by the turbulence and is proportional to the mean wavenumber and to the Mach number of the turbulence

$$M' = u/c = \beta\,(c_0/c)\,M;$$

$\zeta(\mathbf{x}-\alpha\mathbf{V}t, t)$ defines the unit turbulent phase shift as the projection of the direction of the turbulent velocity $\mathbf{m} \equiv \mathbf{u}/u$ on the mean wave-normal direction $\mathbf{n}$ integrated along the ray. The mean phase shift is zero because $\mathbf{n}.\mathbf{m} = 0$, but the deviation from the mean depends on the length $\mathscr{S}$ of the ray.

2.3. *Conservation laws along a ray*

The preceding results were derived from (9*a*), and we now consider (9*b*). Multiply through by Ψ' and use (11*a*, *c*) and (12*a*) to obtain the following alternative form:

$$\Psi'^2(Dk/Dt)+k(D\Psi'^2/Dt) = -c\{\Psi'^2\nabla.\mathbf{k}+\mathbf{k}.\nabla\Psi'^2\}. \tag{15}$$

Introducing the energy density $E \equiv k\Psi'^2$ as the square of the amplitude per unit wave volume, this equation becomes

$$DE/Dt+\nabla.(Ec\mathbf{n}) = 0, \tag{16}$$

which establishes the balance between the energy flux $Ec\mathbf{n}$ radiated relative to the mean flow at the speed of sound in the direction of the wave normal and the energy content of a convected region C. Using (12*b*) this is equivalent to

$$\partial E/\partial t+\nabla.(E\mathbf{w}) = O(E\nabla.\mathbf{u}). \tag{17}$$

This form of the equation applies to the balance of acoustic energy in a domain A at rest; in the case of incompressible turbulence the right-hand side vanishes and the energy flux corresponds to propagation at the group velocity $\mathbf{w}$. The latter was calculated from the dispersion relation (12*a*), which may be expressed in the form $\omega = \omega(\mathbf{k},\mathbf{x},t)$ to exhibit the explicit dependence on the non-uniformity and unsteadiness of the medium. Upon differentiation with regard to position and by use of (11*b*) we have $-\partial\mathbf{k}/\partial t = (\partial\omega/\partial\mathbf{k}).(\partial\mathbf{k}/\partial\mathbf{x})+\partial\omega/\partial\mathbf{x}$; observing that the rate of change of the wave vector along a ray is $d\mathbf{k}/dt = \partial\mathbf{k}/\partial t+(d\mathbf{x}/dt).(\partial\mathbf{k}/\partial\mathbf{x})$, we are thus led to the canonical equations (Hamilton 1827–32)

$$\mathbf{w} \equiv \partial\omega/\partial\mathbf{k} = d\mathbf{x}/dt, \quad d\mathbf{k}/dt = -\partial\omega/\partial\mathbf{x} = -\nabla(\mathbf{k}.\mathbf{u}), \tag{18a, b}$$

in which $\partial\omega/\partial\mathbf{x}$ is evaluated at constant $\mathbf{k}$ from (12*a*). The first equation emphasizes that, although wave fronts propagate at the phase speed $c\mathbf{n}$ relative to the medium, wave 'packets' actually propagate at the group velocity $\mathbf{w}$ [see (12*b*)] relative to a frame at rest, the two being identical only in a quiescent fluid. The wave vector changes along a ray (18*b*) because of turbulent transport; in the linear approximation the latter is calculated for the mean wave only. Introducing in (17) the derivative

$$\mathscr{d}/\mathscr{d}t \equiv \partial/\partial t+\mathbf{w}.\nabla$$

along a ray, we obtain a third form of the equation of energy balance, viz.

$$E^{-1}\mathscr{d}E/\mathscr{d}t+\nabla.\mathbf{w} = 0. \tag{19}$$

Thus the relative change in the energy density along a ray is compensated by variations in the volume occupied by waves; since $E \equiv k\Psi'^2$ the amplitude Ψ' of waves decreases or increases along a ray tube B in inverse proportion to the square root of its cross-section.

For incompressible turbulence in a convected frame we have, from (12*b*),

$$\nabla.\mathbf{w} = c\nabla.\mathbf{n} = c\nabla(\mathbf{k}/k) = (c/k)\nabla.\mathbf{k}+c\mathbf{k}.\nabla(1/k).$$

The term $\nabla.\mathbf{k}$ may be eliminated after substitution in (19) by applying d/dt and using (18*b*): $d(\nabla.\mathbf{k})/dt \simeq \nabla.(d\mathbf{k}/dt) = -\nabla^2(\mathbf{k}.\mathbf{u})$, where we have ignored a nonlinear term. Similarly we omit $\mathbf{u}.\nabla$ in the expression $d/dt = \partial/\partial t + \mathbf{w}.\nabla$ for the total derivative along a ray, i.e. we may take $d/dt \simeq d/dt \equiv \partial/\partial t + (\alpha\mathbf{V}+c\mathbf{n}).\nabla$. Thus the energy equation becomes

$$(d/dt)^2 \ln E - cd^2u_n/ds^2 - cd^2(\ln k)/dt\,ds = O(k_s u), \tag{20}$$

where k_s is the perturbation wavenumber and $u_n \equiv \mathbf{u}.\mathbf{n}$ denotes the component of the turbulent velocity along the wave normal. The arc length per unit time is just the projection of the group velocity $ds/dt = \mathbf{w}.\mathbf{n} = c + u_n$, leading to the approximation $d/dt \sim cd/ds$. Equation (20) therefore reduces to $d^2\{\ln(E/k) - u_n/c\}/ds^2 = 0$, or $\ln \Psi^2 - u_n/c = A_1 s + A_2$, where A_1 are A_2 are constants; taking $s = 0$ at a point in the mean flow (where $u_n = 0$), we have $A_2 = \ln \Psi_0^2$, where Ψ_0 is the incident amplitude and $A_1 = 0$ because the amplitude should remain finite (except at caustics) along the ray as $s \to \infty$. Thus

$$\Psi/\Psi_0 = \exp(\tfrac{1}{2}u_n/c), \quad k/k_0 = 1 - \ln(1 + u_n/c), \tag{21a, b}$$

the relative change in wavenumber being obtained by a similar integration along the ray, between $(k_0, 0)$ and (k, u_n), using (18*b*), viz. $dk/dt = -kdu_n/ds$ in the form $dk/k_0 + du_n/(c+u_n) = 0$. Equations (21*a, b*) are valid only in the linear approximation, so that the relative changes in the amplitude and wavenumber (along the ray) caused by turbulence are $\Psi/\Psi_0 = 1 + \frac{1}{2}u_n/c$ and $k/k_0 = 1 - u_n/c$. Thus the amplitude and wavenumber satisfy $\Psi/\Psi_0 = (k_0/k)^{\frac{1}{2}}$ (the WKB approximation, e.g. Brekhovskikh 1960, p. 195) and, also to a linear approximation, the energy density $E = k\Psi^2$ is conserved, i.e. $E = E_0$ at each position and instant. The relations become trivial in the mean, since $\overline{u_n} = 0$, thus the wavenumber, amplitude, energy density and amplitude squared are all conserved to a linear approximation and to $O(u_n^2/c^2)$.

Thus to a linear approximation there is no net exchange of energy between sound waves and incompressible turbulence; it has been shown in § 2.1 that scattering by a smooth or irregular interface of curvature large on the scale of a wavelength also conserves locally the wave energy. However, in the former case there is no mean reflected wave and the mean-square amplitude is conserved during transmission, whereas in the latter case the amplitude is modified by a factor appearing in (4*b*). This formula applies locally to a flat scattering element lying horizontally, and is extended by (7) to account for a slope in the transmission factor $T_a = \Psi_a \exp(i\Phi_a)$. The effect of incompressible turbulence in the shear layer on the diffraction of very short wavelengths is represented by adding to the scattering phase Φ_a an analogous diffraction term Φ_b given by (14*a*, *b*) and corresponding in the first approximation to the transmission factor (without amplitude variation $\Psi_b = 1$): $T_b = \exp(i\Phi_b)$.

3. The refraction by turbulent shear layers

The transmission coefficients obtained in the previous section can be applied to each wave component of the field generated by any source immersed in the jet, in order to specify the field transmitted through the shear layer into the ambient medium. When the observer and source are both located in the ambient medium, but on opposite sides of a jet, which therefore acts as a noise shield, the transmitted and internal wave fields are described by coupled integral equations, which are solved by means of an operator series, accounting for multiple internal reflexions of all orders. The acoustic power radiated to the far field can be formally expressed in terms of the Fourier spectra received in the ambient medium, the latter including the details of refraction by the system of shear layers.

3.1. *Source located in a semi-infinite jet*

We consider (figure 2*a*) a semi-infinite jet of uniform velocity $\mathbf{V}$ containing a point harmonic source of frequency ω_0 translating at a uniform velocity $\mathbf{U}$ from the position $\mathbf{X} = \mathbf{X}_0$ at time $t = 0$. The sound field in the jet is described by the convected wave equation

$$\{c^{-2}(\partial/\partial t + \mathbf{V}\,.\,\nabla)^2 - \nabla^2\}\,P(\mathbf{X}, t) = S(\nabla, \partial/\partial t)\,\{\delta(\mathbf{X} - \mathbf{X}_0 - \mathbf{U}t)\exp(-i\omega_0 t)\} \tag{22}$$

together with radiation and boundary conditions. The operator $S(\nabla, \partial/\partial t)$ specifies the character of the source, e.g. this operator is $\partial S_0/\partial t$ for a monopole, $\mathbf{S}\,.\,\nabla$ for a dipole, $S_{ij}\,\partial^2/\partial x_i\,\partial x_j$ for a quadrupole, and generally $S_{a_1\ldots a_n}\,\partial_{a_1}\ldots\partial_{a_n}$ with $\partial_a \equiv (\nabla, \partial/\partial t)$ for a multipole of order n. The solution of (22) is a harmonic Green's function which may be convoluted in a Fourier integral with a source of arbitrary spatial and spectral distribution $F(\mathbf{Y}, \omega_0)$ to specify its acoustic pressure field:

$$\mathscr{P}(\mathbf{X}, t) = \int_{-\infty}^{+\infty} F(\mathbf{Y}, \omega_0)\,P(\mathbf{X} - \mathbf{Y}, t; \omega_0)\exp(i\omega_0 t)\,d^3Y\,d\omega_0. \tag{23}$$

The wave generated by the source and incident on the shear layer is given by the particular integral of (22) for an unbounded medium, and this may be determined by Fourier analysis:

$$P_i(\mathbf{X}, t) = \int_{-\infty}^{+\infty} Q_i(\mathbf{k}, \omega)\exp\{i(\mathbf{k}\,.\,\mathbf{X} - \omega t)\}\,d^3k\,d\omega, \tag{24a}$$

where
$$Q_i(\mathbf{k}, \omega) = \frac{S(i\mathbf{k}, -i\omega)}{(2\pi)^3}\frac{\exp(-i\mathbf{k}\,.\,\mathbf{X}_0)\,\delta(\omega - \omega_0 - \mathbf{k}\,.\,\mathbf{U})}{(\omega - \mathbf{k}\,.\,\mathbf{V})^2/c^2 - k^2}. \tag{24b}$$

The poles of $Q_i(\mathbf{k}, \omega)$ may be used to evaluate the k_3-integral in (24*a*) by the theorem of residues, giving

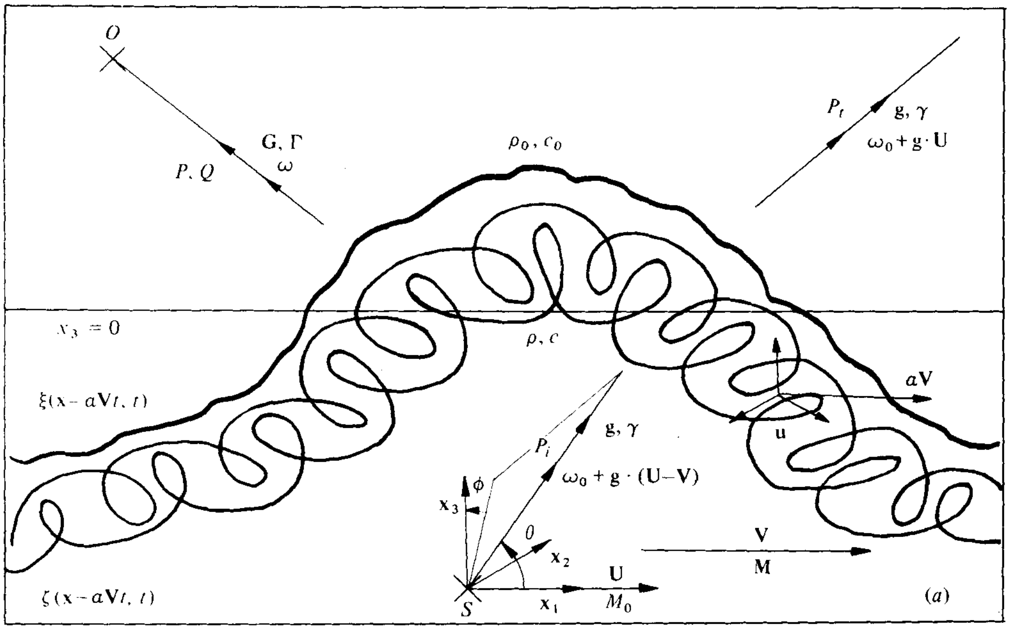
$$P_i(\mathbf{x}, x_3, t) = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} (S/\gamma)\exp\{i\{\mathbf{g}\,.\,(\mathbf{x} - \mathbf{x}_0) + \gamma(x_3 - h) - (\omega_0 + \mathbf{g}\,.\,\mathbf{U})\,t\}\}\,d^2g, \tag{25a}$$

in which

$$\gamma(\mathbf{g}) = \{\{\omega_0 + \mathbf{g}\,.\,(\mathbf{U} - \mathbf{V})\}^2/c^2 - k^2\}^{\frac{1}{2}}, \quad S(\mathbf{g}) \equiv S(-i\mathbf{g}, -i\gamma, i\omega_0 + i\mathbf{g}\,.\,\mathbf{U}) \tag{25b, c}$$

($-h \equiv X_{03}$ denotes the vertical co-ordinate of the source and $x_3 > -h$). The branch of the square root γ chosen for the incident field should be real and positive for waves propagating upwards (in the x_3 direction) and negative and imaginary for evanescent waves. We may exclude the evanescent waves, which decay like $\exp(-|\gamma|h)$, in the specification of the incident wave field just below the shear layer provided that $h \gg \lambda$.

Each plane-wave component $\mathbf{g}$ of the incident wave is modified on transmission by multiplication by the coefficient $A\exp(i\eta)$, which consists of (i) an amplitude



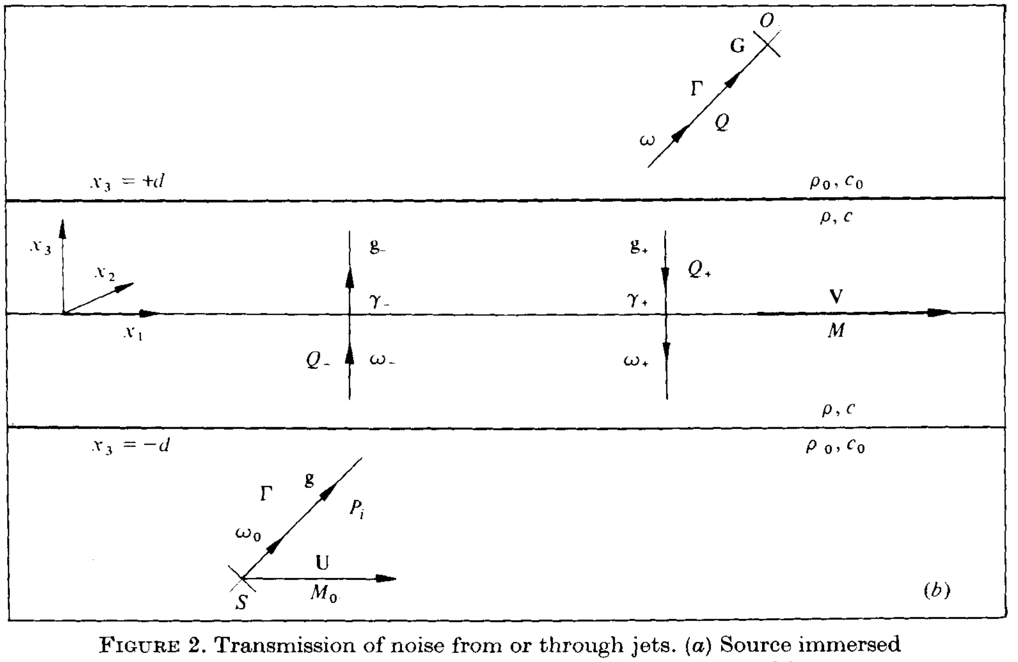


FIGURE 2. Transmission of noise from or through jets. (*a*) Source immersed in a semi-infinite jet. (*b*) Shielding by a jet of finite width.

factor A due to scattering by the interface, given by (4*b*), the effects of the slope (7) being omitted for the sake of brevity, and (ii) a phase shift η, which is caused both by scattering by the interface and by diffraction of rays in the turbulent region (14*a*) Thus

$$A(\mathbf{g}) = 2/\{1+(\rho/\rho_0)(1-M\cos\theta)^2(\Gamma/\gamma)\}, \tag{26a}$$

$$\eta(\mathbf{y},t) = \beta M(c_0/c)\,k\zeta(\mathbf{y},t)+(\gamma-\Gamma)\,\xi(\mathbf{y},t), \tag{26b}$$

where $\mathbf{y} = \mathbf{x}-\alpha\mathbf{V}t$ are co-ordinates moving at mean shear flow velocity and

$$k = \{g^2+[\gamma(\mathbf{g})]^2\}^{\frac{1}{2}}, \quad \Gamma(\mathbf{g}) = \{(\omega_0+\mathbf{g}\,.\,\mathbf{U})^2/c_0^2-g^2\}^{\frac{1}{2}} \tag{26c, d}$$

are the incident wavenumber and the locally transmitted vertical wavenumber, respectively. The transmitted wave just above the shear layer is obtained by formally inserting $A\exp(i\eta)$ in the integrand of (25*a*):

$$P_t(\mathbf{y},x_3,t) = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty}(AS/\gamma)\exp\{i[\mathbf{g}\,.\,\mathbf{y}+\Gamma x_3+\eta(\mathbf{y},t)]-i[\omega_0+\mathbf{g}\,.\,(\mathbf{U}-\alpha\mathbf{V})]\,t\}\,d^2g, \tag{27a}$$

where it is implied that the horizontal wave vector $\mathbf{g}$ is conserved during transmission, in agreement with the expressions (25*b*) and (26*d*) respectively for the incident and locally transmitted vertical wavenumbers. The acoustic pressure $P(\mathbf{y},x_3,t)$ in the ambient medium should be equal to P_t as given by (27*a*) at the instantaneous position of the interface $x_3 = \xi$, or, assuming that this boundary condition can be shifted to the mean plane $x_3 = 0$, we have $P(\mathbf{y},0,t) = P_t(\mathbf{y},x_3 = 0,t)$. The field P in the ambient medium can be represented as a Fourier integral over a spectrum function $Q(\mathbf{K},\omega)$, and in particular, on the plane $x_3 = 0$, P_t is the corresponding space–time Fourier transform of Q, i.e.

$$Q(\mathbf{G},\omega) = \frac{1}{8\pi^3}\int_{-\infty}^{+\infty}P_t(\mathbf{y},t)\exp\{-i\{\mathbf{G}\,.\,\mathbf{y}-(\omega-\alpha\mathbf{V}\,.\,\mathbf{G})\}\,t\}\,d^2y\,dt, \tag{27b}$$

in which $\mathbf{G}$ and ω denote the (received) horizontal component of the wave vector and the frequency, respectively, of the free-space sound field.

Equations (27*a*, *b*) show that the acoustic spectrum Q in the ambient medium may be obtained by applying an integral operator to the source strength S/γ modified by propagation in the jet, i.e. by $Q(\mathbf{G},\omega) = T\{S/\gamma\}$, where

$$T\{f(\mathbf{g})\} \equiv (64\pi^5)^{-1}\int_{-\infty}^{+\infty}f(\mathbf{g})\,A(\mathbf{g})\exp\{i\{(\mathbf{g}-\mathbf{G})\,.\,\mathbf{y}+\eta(\mathbf{y},t) + i\{\omega-\omega_0-\mathbf{g}\,.\,\mathbf{U}+\alpha\mathbf{V}\,.\,(\mathbf{g}-\mathbf{G})\}\,t\}\,d^2g\,d^2y\,dt. \tag{28}$$

This describes the overall effect of transmission through the shear layer, and involves integrations with respect to $\mathbf{g}$ over all the components of the incident wave. For a plane interface $\eta = 0$ and the integrations over space and time give delta functions $\delta(\mathbf{g}-\mathbf{G}) \times\delta(\omega-\omega_*)$, where $\omega_* = \omega_0+\mathbf{g}\,.\,\mathbf{U}$ and the transmission operator degenerates into a multiplication by the transmission factor A [see (26*a*)] evaluated with $\mathbf{g} = \mathbf{G}$ and $\omega = \omega_*$. Otherwise, for a turbulent shear layer (27*a*) is *not* a plane-wave decomposition of the transmitted field, since the phase shift $\eta(\mathbf{y},t)$, given by (26*b*), is a random function of $\mathbf{y}$ and t and thus implies random changes in the frequency and direction of the transmitted waves which are expressed in the spectrum $Q(\mathbf{G},\omega)$ received in the ambient medium through the integral transmission operator T, defined by (28).

3.2. *Jet of finite width as a noise shield*

Consider the case illustrated in figure 2(*b*) of a moving source and an observer at rest both located in the ambient medium on the opposite sides of a two-dimensional jet of uniform velocity $\mathbf{V}$. We assume that the distance between the two shear layers, whose mean planes are $x_3 = \pm d$, is large on the scale of a wavelength ($2d \gg \lambda$), so that the processes of scattering in the upper and lower layers may be considered independently. We have to consider four cases of refraction, each consisting of diffraction in a turbulent region and scattering by an interface.

(i) Transmission across the lower shear layer (with mean plane $x_3 = -d$) from the ambient medium into the jet, described by a lower transmission operator T_- similar to (28) except for the transmission factor A_-, which, as noted in § 2.1, differs from (26*a*) by a factor $(1-M\cos\theta)^{-2}$:

$$T_-\{f(\mathbf{g})\} \equiv T\{f(\mathbf{g})\}_{A_-}, \quad A_- \equiv A\{(1-M\cos\theta)^{-2}\}. \tag{29a}$$

(ii) The field incident on the upper shear layer has become partially incoherent as a consequence of transmission through the lower shear layer, thus the upper transmission operator T_+ involves in addition an integration over a spectrum of incident waves:

$$T_+\{f(\mathbf{g})\} \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} T\{f(\mathbf{g})\}_{\omega_0+\mathbf{g}.\mathbf{U}\equiv\omega}\, d\omega, \tag{29b}$$

i.e. the tone $\omega_0 + \mathbf{g}\,.\,\mathbf{U}$ is formally replaced by a band ω.

(iii) Backscattering at the upper shear layer is described for partially incoherent fields by the upper reflexion operator R_+, which involves the reflexion factor B_+ which appears as an amplitude in (4*a*):

$$R_+\{f(\mathbf{g}_+,\omega_+)\} \equiv \frac{1}{8\pi^3}\int_{-\infty}^{+\infty} f(\mathbf{g}_+,\omega_+)\, B_+(\mathbf{g}_+)\exp(i\Phi_+)\, d^2g_+\, d\omega_+\, d^2y\, dt, \tag{30a}$$

in which

$$\Phi_+(\mathbf{g}_\pm,\omega_\pm) \equiv \{(\mathbf{g}_+-\mathbf{g}_-)\,.\,\mathbf{y}+\eta_+(\mathbf{y},t)\}+\{\omega_+-\omega_-+(1-\alpha)\,\mathbf{V}\,.\,(\mathbf{g}_--\mathbf{g}_+)\}\,t. \tag{30b}$$

(iv) Backscattering at the lower shear layer is described by the lower reflexion operator R_-, given by an expression similar to (30*a*) with $\pm$ signs interchanged (also for Φ_-) and with $B_- \equiv B_+\{(1-M\cos\theta)^{-2}\}$.

For upper (i, iii) and lower (ii, iv) scattering the height of the interface in the phase shift (26*b*) is respectively $\pm d + \xi_\pm$.

These four integral operators may be used to describe the fields both inside and above the jet due to a source below in the ambient medium. The waves propagating upwards in the jet, Q_+ say, are a result of transmission through the lower shear layer from the source *and* of lower reflexion at the shear layer of downward-travelling waves, Q_- say, i.e.

$$Q_+(\mathbf{g}_+,\omega_+) = T_-\{(S/\gamma)_\mathbf{g}\} + R_-\{Q_-(\mathbf{g}_-,\omega_-)\}. \tag{31a}$$

Similarly the upward-propagating waves Q_+ generate by reflexion at the upper shear layer the downward-travelling waves Q_-, i.e.

$$Q_-(\mathbf{g}_-,\omega_-) = R_+\{Q_+(\mathbf{g}_+,\omega_+)\}, \tag{31b}$$

and the field radiated to the observer is given by

$$Q(\mathbf{G},\omega) = T_+\{Q_+(\mathbf{g}_+,\omega_+)\}. \tag{32}$$

In order to determine Q from (32), we have to solve the pair of coupled integral equations (31 a, b) for the fields in the double layer, generalizing those of Howe (1976). Equations (31 a, b) resemble respectively integral equations of the first and second kinds (Whittaker & Watson 1927, chap. xi) coupled by an interchange of the integrands and uncoupled by substitution of one in the other, e.g. for Q_+

$$Q_+(\mathbf{g}_+, \omega_+) = T_-\{(S/\gamma)_{\mathbf{g}}\} + R_- R_+\{Q_+(\mathbf{g}_+, \omega_+)\}. \tag{33}$$

This integral equation of the second kind states that upward-propagating waves in the jet arise from the transmission of waves from the source plus *double* reflexions of themselves at the upper and lower shear layers. Equation (32) may be solved by iteration (Liouville 1837–8); using the symbolic notation for operator series $(1-R_- R_+)^{-1} \equiv 1 + R_- R_+ + R_- R_+ R_- R_+ + \dots$ and (32) leads to the following operational form of the transmitted Fourier field:

$$Q(\mathbf{G}, \omega) = S\{(S/\gamma)_{\mathbf{g}}\}, \quad S \equiv T_+(1-R_- R_+)^{-1} T_-. \tag{34a, b}$$

Equation (34b) defines the overall refraction operator S, which should be read in order of application from right to left and consists of a transmission from the source to the jet followed by multiple reflexions of all orders $n = 0, \dots, 2N, \dots,$ within the latter, and transmission from the jet to the ambient medium.

The nth-order term in (33b), $T_+(R_- R_+)^n T_-$, is an integral of $11 + 12n$ dimensions in the fields, corresponding (as will be shown in § 3.3) to $26 + 24n$ dimensions for the intensity. The terms which may be necessary to determine to a specified accuracy the sound field transmitted through a double shear layer may be expressed using the integral operators (28)–(30) much in the same way as in the zero-order approximation, which is $Q_0(\mathbf{G}, \omega) = D\{(S/\gamma)_{\mathbf{g}}\}$, where $D \equiv T_+ T_-$ and involves successive transmissions at the lower and upper shear layers. From (29 a, b) follows the double transmission operator for a turbulent shear-layer shield:

$$\begin{aligned} D\{f(\mathbf{g})\} \equiv \{4(2\pi)^{11}\}^{-1} & \int_{-\infty}^{+\infty} f(\mathbf{g})\, A_-(\mathbf{g})\, A(\mathbf{g}_+) \\ & \times \exp\{i\{(\mathbf{g}-\mathbf{g}_+)\,.\,\mathbf{y}_- + \eta_-(\mathbf{y}_-, t_-)\} + i\{\omega_+ - \omega_0 - \mathbf{g}\,.\,\mathbf{U} + \alpha \mathbf{V}\,.\,(\mathbf{g}-\mathbf{g}_+)\}t_-\} \\ & \times \exp\{i\{(\mathbf{g}_+ - \mathbf{G})\,.\,\mathbf{y}_+ + \eta_+(\mathbf{y}_+, t_+)\} \\ & + i\{\omega - \omega_+ + \alpha \mathbf{V}\,.\,(\mathbf{g}_+ - \mathbf{G})\}t_+\}\, d^2g\, d^2y_-\, dt_-\, d^2g_+\, d\omega_+\, d^2y_+\, dt_+, \end{aligned} \tag{35}$$

in which the integrations reveal that the coherent field d^2g incident on the lower layer $d^2y_- dt_-$ has become partially incoherent after the first transmission $d^2g_+ d\omega_+$, the incoherence being increased by the second transmission across the upper shear layer $d^2y_+ dt_+$.

3.3. *Power flux of the radiation field*

In both the single-layer problem (3.1) and the double-layer problem (3.2) we have assumed the ambient medium to be at rest, the jet to have velocity $\mathbf{V}$ and the source to be in motion at velocity $\mathbf{U}$. The results, such as (28) and (35), will also apply when the ambient medium is in uniform motion at velocity $\mathbf{V}_0$, provided that the co-ordinate system is assumed to move at this velocity so that $\mathbf{y} \to \mathbf{y} - \mathbf{V}_0 t$ and the jet and source velocities become respectively $\mathbf{U} - \mathbf{V}_0$ and $\mathbf{V} - \mathbf{V}_0$. The effects of the motion of the ambient medium can be included in the intensity of radiation by means of Doppler factors $\mathscr{M}_0 = 1 - (V_0/c_0)\cos\theta$, and we need consider only the case when the ambient medium is at rest. The transmitted acoustic power W is defined as the flux of energy Pv_3 across a horizontal plane (x_3 = constant > 0) situated above the shear layer, averaged for all time:

$$W = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{+\tau} \int_{-\infty}^{+\infty} P(\mathbf{y}, t)\, v_3(\mathbf{y}, t)\, d^2y\, dt. \tag{36}$$

The pressure in the ambient medium is the Fourier transform of the acoustic spectrum (or observer Fourier field) Q, i.e.

$$P(\mathbf{y}, x_3, t) = \int_{-\infty}^{+\infty} Q(\mathbf{G}, \omega) \exp\{i(\mathbf{G}\,.\,\mathbf{y} + \Gamma x_3 - \omega t)\}\, d^2G\, d\omega, \tag{37a}$$

and on substitution of (28) and (35) we obtain the explicit wave fields in the ambient medium for the single and double shear layers respectively. In the latter case the higher-order approximations, which involve multiple reflexions, can also be included [by means of (34*a*, *b*)] in the spectrum Q, which describes the effects of refraction for any specific system of shear layers, whether single, double or multiple, and thus the function Q may be used in a general expression for the acoustic power W. The latter is given by (36) in terms of the pressure P and the vertical component of the acoustic velocity v_3, which are related in free space by the equation of momentum $\rho_0\, \partial v_3/\partial t + \partial p/\partial x_3 = 0$, i.e. for each wave component $\mathscr{M}_r \omega v_3 = (\Gamma/\rho_0)\, P$, where the Doppler factor $\mathscr{M}_r = 1 - M_0 \cos\theta$ accounts for the motion of the source with Mach number $M_0 = U/c_0$. Thus v_3 is given by

$$v_3(\mathbf{y}, x_3, t) = \int_{-\infty}^{+\infty} Q(\mathbf{G}, \omega)\, (\Gamma/\omega M_r \rho_0) \exp\{i(\mathbf{G}\,.\,\mathbf{y} + \Gamma x_3 - \omega t)\}\, d^2G\, d\omega. \tag{37b}$$

Substitution of (37*a*, *b*) in (36) gives

$$W = (\rho_0 \mathscr{M}_r)^{-1} \operatorname{Re}\left\{\lim_{\tau\to\infty} \frac{1}{4\tau} \int_{-\tau}^{+\tau} \int_{-\infty}^{+\infty} (\Gamma'^*/\omega')\, Q(\mathbf{G}, \omega)\, Q^*(\mathbf{G}', \omega') \right.$$
$$\left. \times \exp\{i\{(\mathbf{G} - \mathbf{G}')\,.\,\mathbf{y} + (\Gamma - \Gamma'^*)\, x_3 - (\omega - \omega')\, t\}\}\, d^2G'\, d\omega'\, d^2G\, d\omega\, d^2y\, dt\right\}, \tag{38}$$

where an asterisk denotes a complex conjugate and $\mathbf{G}'$ and ω' are marked with primes to distinguish them from $\mathbf{G}$ and ω. The spatial integration is an example of the Fourier integral property

$$\int_{-\infty}^{+\infty} f(\mathbf{G}') \exp\{i(\mathbf{G}' - \mathbf{G})\,.\,\mathbf{y}\}\, d^2y\, d^2G' = (2\pi)^2 f(\mathbf{G}). \tag{39}$$

The temporal averaging gives the continuous Kronecker delta function, defined as unity if the variables are equal and zero if they are unequal:

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{+\tau} \exp\{i(\omega' - \omega)\, \tau\}\, d\tau = \delta_{\omega\omega'}. \tag{40}$$

Equations (39) and (40) simplify (38), which vanishes identically when Γ is pure imaginary, showing that evanescent waves do not radiate energy to the far field, i.e. the range of integration in wave-vector (or wavenumber and frequency) space is determined by the (propagating) incident and transmitted modes only:

$$W = \frac{2\pi^2}{\rho_0 \mathscr{M}_r} \int^{\operatorname{Re}\gamma,\, \Gamma} (\Gamma/\omega)\, Q(\mathbf{G}, \omega)\, Q^*(\mathbf{G}, \omega')\, \delta_{\omega\omega'}\, d\omega'\, d^2G\, d\omega. \tag{41}$$

Choosing spherical polar co-ordinates (figure 2*a*) with origin at the source, the axis $\theta = 0$ parallel to the x_1 axis (i.e. parallel to the velocity of the source **U**, which is parallel to **V**) and $\phi = 0$ in the 'fly-over' plane (perpendicular to the shear layer's mean plane $x_3 = 0$) allows the received wave vector to be expressed as

$$\mathbf{K} \equiv (\mathbf{G}, \Gamma) = (\omega/c_0)(\cos\theta, \sin\theta\sin\phi, \sin\theta\cos\phi).$$

The (x_1, x_2) wave-vector element $d^2G = (\partial G_1/\partial\theta)(\partial G_2/\partial\phi)\, d\theta\, d\phi$ can be written (Howe 1975) in terms of the solid-angle element $d\Omega \equiv \sin\theta d\theta d\phi$, which is integrated over the half of the unit sphere specified by $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$, which we denote by $\mathscr{U}$:

$$W = (2\pi^2/\rho_0 c_0^3)\int^{\mathscr{U}} \{(\sin^2\theta\cos^2\phi)/(1-M_0\cos\theta)\} \times \int^{\mathrm{Re}\gamma,\,\Gamma} \omega^2 Q(\mathbf{G},\omega)\,\delta_{\omega\omega'}\,Q^*(\mathbf{G},\omega')\,d\omega' d\Omega\, d\omega. \quad (42)$$

The acoustic power can be made explicit by substituting for Q and Q^* for the shear-layer system being considered, e.g. using (28) for single and (35) [or (34*a*, *b*)] for double shear layers, the $d\omega d\omega'$ integrations being trivial in the case of a plane interface $\eta = 0$. In the presence of turbulence or irregularities, the incoherent phase term $\exp\{i[\eta(\mathbf{y},t) - \eta(\mathbf{y}',t')]\}$ depends on the particular realization of the shear layer, or, when averaged, on its correlation properties.

4. The statistics of energy propagation

Although it has been shown (in § 1) that the refraction of sound in a turbulent shear layer conserves energy to a linear approximation, the correlation of phase shifts evinces a nonlinear effect of attenuation of the transmitted wave. This is associated with part of the acoustic energy being absorbed or re-directed by the corresponding components of the turbulence spectrum, the resultant attenuation being compensated for partially by the interference between correlated components of the refracted wave. These random effects are expressed by means of statistical scales appearing in a characteristic function for interference and an autocorrelation coefficient for turbulent diffraction, which appear to be consistent with the experiments of Schmidt & Tilmann (1970) and Ho & Kovasznay (1976*a*, *b*), respectively, and complete the specification of the energy field transmitted into the ambient medium.

4.1. *Characteristic attenuation and interference*

The random phase shifts (26*b*) resulting from the transmission of sound through a shear layer are of the form $\exp\{i\mu(\mathbf{g})\,\upsilon(\mathbf{y},t)\}$, both for scattering by the irregular interface, with $\mu \equiv \gamma - \Gamma$ and $\upsilon \equiv \xi$, and for diffraction by the turbulent region, with $\mu \equiv \beta Mk(c_0/c)$ and $\upsilon \equiv \zeta$. The acoustic power, being quadratic, involves also $\mu' \equiv \mu(\mathbf{g}')$, which is evaluated at a different wavenumber $\mathbf{g}'$, corresponds to a distinct event $\upsilon' \equiv \upsilon(\mathbf{y}',t')$ and gives rise to a phase difference that may be averaged over all possible realizations of the shear layer:

$$C(\mu,\mu') = \langle \exp\{i\{\mu\upsilon(\mathbf{y},t) - \mu'\upsilon(\mathbf{y}',t')\}\}\rangle. \quad (43)$$

From the point of view of the mathematical theory of statistics (Kolmogorov 1950) we must consider a bivariate random process defined by the phases υ and υ' for which

(43) is the joint characteristic function (von Mises 1960). The latter describes completely the statistics of the process since, when expanded in a double series, it gives

$$C(\mu,\mu') = \sum_{n,m=0}^{\infty} i^{n+m}\frac{\mu^n\mu'^m}{n!m!}M_{nm}, \quad M_{nm} \equiv \langle\{v(\mathbf{y},t)\}^n\{v(\mathbf{y}',t')\}^m\rangle, \qquad (44a,b)$$

where the M_{nm} define the moments of the distribution.

The first-order moment is the mean value $M_{01} = \langle v\rangle$, which is identically zero for either the displacement of the interface or the random phase shift along a ray passing through the turbulence since $\langle\xi\rangle = 0 = \langle\zeta\rangle$. The second-order moment $M_{11} = \langle vv'\rangle$ for a stationary random process depends on only the spatial and temporal separations $\mathbf{z} \equiv \mathbf{y}-\mathbf{y}'$ and $\tau \equiv t-t'$ and is known as the autocorrelation function

$$D(\mathbf{z},\tau) = \langle v(\mathbf{y},t)\,v(\mathbf{y}+\mathbf{z},t+\tau)\rangle. \qquad (45a)$$

The latter can be normalized with regard to the other second-order moment, namely the variance $M_{20} = \sigma^2$, to define the autocorrelation coefficient

$$E(\mathbf{z},\tau) \equiv \sigma^{-2}D(\mathbf{z},\tau), \quad \sigma^2 \equiv D(0,0) = \langle\{v(\mathbf{y},t)\}^2\rangle, \qquad (45b,c)$$

where σ is a root-mean-square value, which vanishes only for a plane interface devoid of turbulence. If the realizations of the shear layer are statistically symmetric about the mean plane all the moments whose order $N = n+m$ is odd vanish, and those of even order, which satisfy $M_{nm} = M_{mn}$ and thus are $\frac{1}{2}N(N+1)$ in number, are proportional to σ^N. In order to gain a preliminary idea of the form of the characteristic function for small r.m.s. values, (43) may be expanded in a single series, and is consistent to $O(\sigma^4)$ with the expression

$$C(\mu,\mu') = \exp\{-\tfrac{1}{2}\sigma^2[\mu^2+\mu'^2-2\mu\mu'E(\mathbf{z},\tau)]\}. \qquad (46)$$

This characterizes the sound attenuation $\exp\{-\frac{1}{2}\sigma^2(\mu^2+\mu'^2)\}$ and the amplification effect $\exp\{\sigma^2\mu\mu'E\}$ of interference between correlated diffracted waves, which can compensate partially for the loss of energy $\mu^2+\mu'^2 > 2\mu\mu'E$ for separate events $E < 1$.

In order to identify the random process corresponding to a given characteristic function we note that from the definition (43) the latter is the Fourier transform in (vv') of the probability density function $F(v,v')$. Applying the Fourier inversion theorem in (μ,μ') to (46), we find

$$F(v,v') = \frac{1}{2\pi\sigma^2}\frac{1}{(1-E^2)^{\frac{1}{2}}}\exp\left\{-\frac{v^2+v'^2-2vv'E}{2\sigma^2(1-E^2)}\right\}, \qquad (47)$$

i.e. the probability density function of a bivariate Gaussian or normal random process. This result, that the propagation of sound in a turbulent shear layer may be approximated by a normal process, may be established if we consider a sequence of realizations τ_n separated by more than a correlation time, $\tau_n-\tau_{n-1} > L_0$, to be statistically independent. Using the notation of a univariate process, in the interests of simplicity of writing, if the probability density of the nth realization of the shear layer is $F_n(v)$, the overall probability density function

$$F_N(v) = \prod_{n=1}^{\mathcal{N}} F_n(v)$$

will converge to a Gaussian function as $N \to \infty$, according to the central limit theorem, if and only if Lindeberg's (1922) condition is met, i.e.

$$\lim_{N\to\infty} \frac{1}{\sigma_N} \sum_{n=1}^{N} \int_{|v_n|>\epsilon\sigma_N} v^2 F_n(v)\, dv = 0. \tag{48}$$

If the shear layer contains turbulence and/or irregularities, i.e. $\infty > \bar{\sigma} > \sigma_n > \tilde{\sigma} > 0$, the total variance

$$\sigma_N^2 = \sum_{n=1}^{N} \sigma_n^2$$

and the r.m.s. deviation $\sigma_N > N^{\frac{1}{2}}\tilde{\sigma}$ both tend to infinity as $N \to \infty$. Large phase shifts are improbable. Let there exist a b such that $F_n(v) < \Delta/N$ for some Δ and all $|v| > b$; then for (any) sufficiently large N such that $\epsilon\sigma_N > \epsilon N^{\frac{1}{2}}\tilde{\sigma} > b$, the sum of integrals in (48) has an upper bound $\bar{\sigma}^2\Delta$, and the condition is met. Also for long time spans $\Delta\tau > NL_0$ as the autocorrelation tends to zero, according to the ergodic theorem:

$$\lim_{\tau\to\infty} E(\tau) = 0, \qquad \langle f(v)\rangle = \overline{f(t)}, \tag{49a, b}$$

$$\langle f(v)\rangle \equiv \int_{-\infty}^{+\infty} f(v)\, F(v)\, dv, \quad \overline{f(t)} \equiv \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{+\tau} f(v(t))\, dt. \tag{49c, d}$$

Thus the preceding statements apply equivalently (Khinchin 1948) to averages over all realizations (denoted by angle brackets) and over all time (denoted by overbars).

The physical and mathematical arguments (46) and (47), respectively, for the propagation of sound in irregular or turbulent shear layers to be a normally distributed random process bear comparison (figure 3*a*) with the experiments of Schmidt & Tilmann (1970). Their measurements of the phase shift of sound transmitted through strong turbulence generated by a thick rod in a mean flow and through weak turbulence further downstream in the wake of a thinner rod agree approximately with a Gaussian probability density.

4.2. *Anisotropic and unsteady autocorrelation*

We have assumed that the autocorrelation of the phase shift vanishes as the separation tends to infinity and becomes negligible for events delayed by more than a finite correlation time. Although these properties seem natural enough in the presence of turbulence, they are further elucidated by the consideration of (45*a*), say for the total phase shift due to turbulent diffraction, which is given by (14*a, b*) as the projection of the turbulent Mach vector $\mathbf{M} \equiv \mathbf{u}/c$ on the mean wave vector $\mathbf{k}$ integrated along the arc length ds of a ray. It will be shown in §4.3 that only individual wave components need be considered, thus

$$D(\mathbf{z}, \tau) = \int_{\mathscr{S}} k_i\, k_j\, M_{ij}(s\mathbf{n} + \mathbf{z}, s/c + \tau)\, \delta_{ss'}\, ds\, ds', \tag{50}$$

where $\delta_{ss'}$ is the Kronecker delta function, $\mathbf{n} \equiv \mathbf{k}/k$ the wave normal and $M_{ij} \equiv \langle M_i\, M_j\rangle$ the autocorrelation of the turbulent Mach vector. M_{ij} is the Fourier transform of a

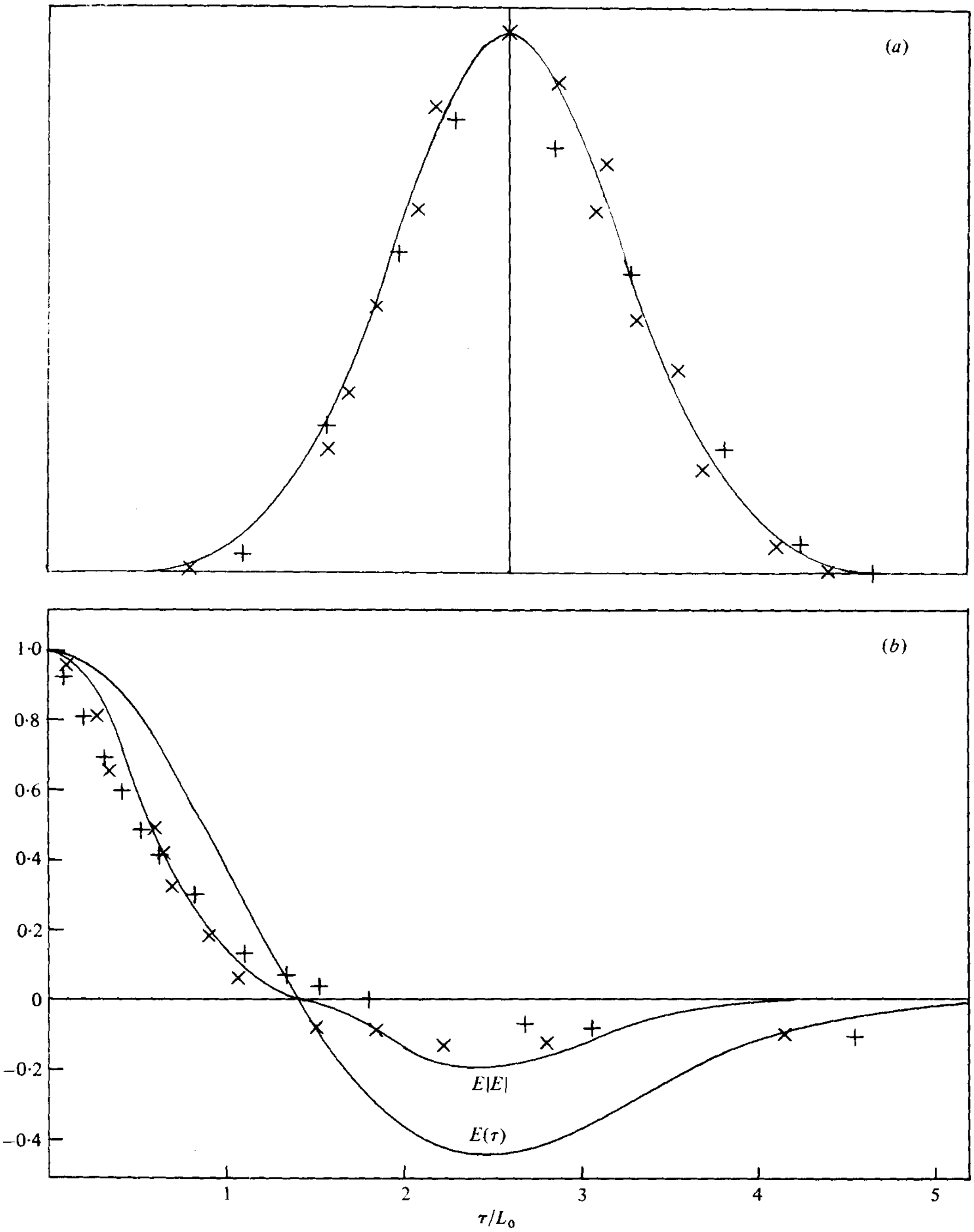


FIGURE 3. Statistics of the acoustic phase shift. (*a*) Probability density function: ——, normal or Gaussian distribution; +, weak wake turbulence (Schmidt & Tilman 1970); ×, strong wake turbulence (Schmidt & Tilmann 1970). (*b*) Autocorrelation coefficient: ——, theoretical formula (56) deduced in § 4.2; +, frequency of test pulse $f = 10$ kHz (Ho & Kovasznay 1976*a*, *b*); ×, $f = 20$ kHz (Ho & Kovasznay 1976*a*, *b*).

turbulence spectrum $N_{ij}(\boldsymbol{\kappa}, \chi)$ (Batchelor 1953, chap. 2). We may expand M_{ij} in power series in the spatial separation $\mathbf{z}$ and time delay τ,

$$M_{ij}(s\mathbf{n}+\mathbf{z}, s/c+\tau) = \sum_{p,q=0}^{\infty} \frac{i^{p+q}}{p!\,q!} \int_{-\infty}^{+\infty} (\boldsymbol{\kappa}\,.\,\mathbf{z})^p (\chi\tau)^q \exp\{is(\mathbf{n}\,.\,\boldsymbol{\kappa}-\chi/c)\} N_{ij}(\boldsymbol{\kappa},\chi)\, d^3\kappa\, d\chi, \tag{51}$$

since the problem involves a large parameter, viz. the length of the ray measured on the scale of a wavelength.

On substitution of (51) in (50) the ds integration may be extended to $(-\infty, +\infty)$, because M_{ij} vanishes outside the turbulent region, and gives a Dirac delta function $\delta(\mathbf{n}\,.\,\boldsymbol{\kappa}-\chi/c)$, which shows that the spectral components of turbulence whose phase speed in the wave-normal direction equals the speed of sound $\chi/\mathbf{n}\,.\,\boldsymbol{\kappa} = c$ are responsible for the diffraction of acoustic waves. We may thus introduce the reduced turbulence spectrum in the wave-normal direction:

$$N(\boldsymbol{\kappa}) \equiv n_i n_j N_{ij}(\boldsymbol{\kappa}, c\mathbf{n}\,.\,\boldsymbol{\kappa}), \tag{52}$$

whose integral over the turbulence wave-vector space $\mathscr{W}$ appears in the zero-order term in D, i.e. the variance of the total phase shift

$$D(0,0) = ck^2 \int_{\mathscr{S}} \int_{\mathscr{W}} N(\boldsymbol{\kappa})\, d^3\kappa\, ds = \{\beta M(c_0/c)\, k\}^2 \langle \zeta^2 \rangle, \tag{53a, b}$$

which specifies, noting (14a), the variance of the unit phase shift. We assume N to be an even function (e.g. because $D(\mathbf{z}, t)$ is a statistically stationary function), i.e. $N(\boldsymbol{\kappa}) = N(-\boldsymbol{\kappa})$, in which case all terms of odd order in p or q vanish and the lowest-order non-zero terms in the expansion [after $p = 0 = q$; see (53)] are $p = 2$ and $q = 0$ and vice versa, yielding

$$L_1^{-2}, L_2^{-2}, L_0^{-2} = \int_{\mathscr{S}} \int_{\mathscr{W}} \{\kappa_1^2, \kappa_2^2, (c\mathbf{n}\,.\,\boldsymbol{\kappa})^2\} N(\boldsymbol{\kappa})\, d^3\kappa\, ds \Big/ \int_{\mathscr{S}} \int_{\mathscr{W}} N(\boldsymbol{\kappa})\, d^3\kappa\, ds, \tag{54}$$

in which L_1 and L_2 have the dimensions of length and L_0 those of time. If the turbulence is isotropic, i.e. $N_{ij}(\boldsymbol{\kappa}, \chi) = A(\kappa)\,(\kappa_i\kappa_j - \kappa^2\delta_{ij})$, then $N(\boldsymbol{\kappa}) = A(\kappa)\,\{(\boldsymbol{\kappa}\,.\,\mathbf{n})^2 - \kappa^2\}$, which averaged over a sphere $|\boldsymbol{\kappa}| =$ constant is independent of $\mathbf{n}$, as are (53a) and (54). The quantities L_1, L_2 and L_0 vary inversely with the κ_1 and κ_2 components of the turbulence wave vector and the frequency of the turbulence χ, respectively, and are thus large in weak turbulence and smaller in stronger turbulence, characterizing the scales over which significant diffraction occurs. These quantities would be infinite only in the absence of turbulence, and generally define three scales of a turbulent and irregular shear layer: the refraction time L_0 and the longitudinal and transverse refraction lengths L_1 and L_2.

On substituting (53a) and (54) in (51) and neglecting terms $O(z^4, z^2\tau^2, \tau^4)$, the three-dimensional autocorrelation coefficient for general anisotropic and unsteady turbulence may be written in the form

$$E(\mathbf{z}, \tau) = E(z_1/L_1)\, E(z_2/L_2)\, E(\tau/L_0), \tag{55}$$

which consists of factors involving respectively the longitudinal and transverse separation and the time delay (z_1, z_2 and τ) and the corresponding refraction lengths and time (L_1, L_2 and L_0). The one-dimensional autocorrelation $E(z/L)$ is the Fourier transform of the reduced turbulence spectrum, e.g. if the latter is Gaussian, i.e. $N(\kappa) = (L/2\sqrt{\pi}) \exp(-\frac{1}{4}\kappa^2 L^2)$, then $E(z/L) = \exp(-z^2/L^2)$. This negative-exponential form (Chernov 1967, chap. 1) ensures that the correlation decays to zero as $z \to \infty$ and becomes negligible for separations larger than the refraction scale. A whole family of autocorrelations can be obtained from the basic form by applying a polynomial p of dimensionless derivatives $L\partial/\partial L$, e.g., for $p(L\partial/\partial L) = 1 - L\partial/\partial L$, we obtain the autocorrelation coefficient

$$E(z/L) = (1 - 2z^2/L^2) \exp(-z^2/L^2), \tag{56}$$

which has the property $$\int_{-\infty}^{+\infty} E(z/L)\, d(z/L) = 0.$$

The analogous expression for diffraction by turbulence states that negative and positive values of the acoustic phase shift balance over the whole shear layer, and for scattering by an irregular interface it implies that

$$\int_{-\infty}^{+\infty} \xi(z)\, dz = 0,$$

i.e. the volume occupied by the jet (or the ambient medium) is constant.

The spatial autocorrelation coefficient of the phases has been measured perpendicular to the direction of propagation of radio waves in the atmosphere by Tatarski (1967, chap. 5), who plotted an empirical curve through his experimental results, which showed that the correlation became negative beyond a certain separation. The time correlations of phase shifts of sound transmitted through a jet issuing from a 12 cm nozzle were measured by Ho & Kovasznay (1976*a*, *b*); their experimental points for frequencies of 10 and 20 kHz have been reproduced in figure 3(*b*) for comparison with the theoretical formula (56). The experimental conditions correspond to a double transmission through both shear layers of a jet, so that the correlation is weaker, viz. for *two* statistically independent refractions the autocorrelation is approximately the product of correlations with unchanged sign, i.e. $E|E|$, a prediction quantitatively consistent with experiment (see figure 3*b*). Both the two theoretical curves and the two sets of experimental results, which involve the refraction time L_0 defined respectively by (54) and by the scaling indicated in Ho & Kovasznay (1976), show that the correlation reduces from unity to zero at a time delay of approximately $\tau \simeq L_0/\sqrt{2}$ and is negative thereafter.

4.3. *Concept of spectral directivity*

The statistical description of sound propagation in a shear layer is summarized formally by the characteristic function (46) for Gaussian phase fluctuations, which involves the autocorrelation coefficient E and applies in the following cases: (i) diffraction by turbulence along a ray of mean length $\mathscr{S}$ in the turbulent region increased for oblique incidence at an angle $\theta \neq \frac{1}{2}\pi$ to the jet velocity by a factor $\operatorname{cosec}\theta$, so that the variance, given by (53*b*), can be written in the form $\sigma^2 \equiv b^2 \operatorname{cosec}\theta$, where b defines the effective thickness of the shear layer; (ii) scattering by an irregular and unsteady interface of root-mean-square height (or displacement) $\sigma \equiv a$. If we consider one wave component, an assumption to be justified in (58) and (59), then $\mu \equiv \mu'$ and (46) reduces to $\exp\{-\sigma^2\mu^2(1-E)\}$, where $\mu \equiv \beta M\ (c_0/c)\, k$ for diffraction in turbulence and $\mu \equiv \gamma - \Gamma$ for scattering by an interface. The two refraction processes are assumed to be statistically independent because the former involves the local turbulent velocity perturbations within the shear layer whereas the latter depends on the outward shape of the mixing layer acting as a boundary of the jet. Thus the product of their characteristic functions gives the overall characteristic function for refraction by a shear layer:

$$C(\mathbf{z}, \tau) = \exp\{-Q[1 - E(\mathbf{z}, \tau)]\}, \tag{57a}$$

$$Q \equiv a^2(\gamma - \Gamma)^2 + \beta^2 M^2 (c_0/c)^2 k^2 b^2 \operatorname{cosec}\theta, \tag{57b}$$

where the attenuation factor is equal to the square of the r.m.s. height of the irregularities times the difference in the vertical wavenumbers plus the square of the effective thickness of the shear layer times the wavenumber $k_u = ku/c$ for turbulent diffraction. The latter term also includes a factor cosec θ, showing that the attenuation caused by the turbulence increases towards grazing directions, being present at all angles. Conversely, the attenuation associated with scattering by the irregularities of the interface increases away from the direction for which $\gamma = \Gamma$, i.e. the wave component which is transmitted without deflexion is not attenuated.

Since it has been shown (in §§2.1 and 2.3) that the processes of scattering by irregular interfaces and diffraction by turbulence conserve energy locally to a linear approximation, it is emphasized that the effects described by (57a, b), which were derived by means of a statistical analysis, are nonlinear. Even for turbulence which is effectively incompressible ($\beta^2 M^2 \ll 1$) the corresponding attenuation [the second term in (57b)] can be non-negligible if the thickness of the shear layer is large on the scale of a wavelength, i.e. $k^2b^2 = (2\pi b/\lambda)^2 \gg 1$. Similarly (away from the direction of undeflected transmission $\gamma = \Gamma$) the attenuation associated with scattering by the interface is significant if the r.m.s. height of irregularities is large on the scale of a wavelength, i.e. $k^2a^2 = (2\pi a/\lambda)^2 \gg 1$. The attenuation $\exp(-Q)$ can be realized only for uncorrelated events which are separated or delayed by more than a refraction length *or* time ($z_1 > L_1$ or $z_2 > L_2$ or $\tau > L_0$); otherwise, for correlated events ($z_1 < L_1$ and $z_2 < L_2$ and $\tau < L_0$) the preceding attenuation cannot be achieved, because (57a) involves $\exp(+QE)$, which appears as an effective amplification by interference between correlated components of the refracted wave, which preserves *some* of the energy for separate events since $E < 1$.

The statistics of refraction (which were assumed stationary) depend on the two events $(\mathbf{y}, t)$ and $(\mathbf{y}', t')$ only through the separation and delay $\mathbf{z} = \mathbf{y} - \mathbf{y}'$ and $\tau = t - t'$, and these may be taken as dummy variables instead of $(\mathbf{y}, t)$ when performing the substitution of the fields, e.g. (28), in the expression (42) for the acoustic power, viz.

$$\begin{aligned} W = {} & \{2\pi^2/\rho_0 c_0^3 (64\pi^5)^2\} \operatorname{Re} \int_{\mathscr{U}} \{\sin^2\theta \cos^2\phi/(1 - M_0 \cos\theta)\} \\ & \times \int_{-\infty}^{+\infty} \omega^2 (AS/\gamma) \exp\{i(\mathbf{g} - \mathbf{G}) . \mathbf{z} + i\{\omega - \omega_0 - \mathbf{g} . \mathbf{U} + \alpha \mathbf{V} . (\mathbf{g} - \mathbf{G})\}\tau\} \\ & \times (AS/\gamma)'^* \exp\{i(\mathbf{g} - \mathbf{g}') . \mathbf{y}' + i\{\omega - \omega' - (\mathbf{g} - \mathbf{g}') . (\alpha \mathbf{V} - \mathbf{U})\} t'\} \\ & \times \delta_{\omega\omega'} C(\mathbf{z}, \tau)\, d^2g'\, d^2y'\, dt'\, d^2g\, d^2z\, d\tau\, d\omega'\, d\Omega\, d\omega. \end{aligned} \tag{58}$$

The result (39) permits six of the integrations to be performed trivially, by setting $\mathbf{g} = \mathbf{g}'$ and $\omega = \omega'$, justifying our consideration of a single wave component in (57).

No further trivial integrations are generally possible for a turbulent or irregular shear layer and we have

$$W = \{8(2\pi)^5 \rho_0 c_0^3\}^{-1} \int_{\mathscr{U}} \{\sin^2\theta\cos^2\phi/(1-M_0\cos\theta)\} \int^{\mathrm{Re}\gamma,\,\Gamma} (AS\omega/\gamma)^2 C(\mathbf{z},\tau)$$
$$\times \exp\{i(\mathbf{g}-\mathbf{G}).\mathbf{z}+i\{\omega-\omega_0-\mathbf{g}.\mathbf{U}+\alpha\mathbf{V}.(\mathbf{g}-\mathbf{G})\}\tau\}\, d^2g\, d^2z\, d\tau\, d\Omega\, d\omega. \tag{59}$$

Formula (39) would be applicable again only in the case of a plane interface devoid of turbulence ($a = 0 = b$), when there is no attenuation or interference ($c = 1$) and in which case we may write

$$\{dW/d\Omega\, d\omega\}_0 = \{8\pi^2\rho_0 c_0(1-M_0\cos\theta)\}^{-1}\delta(\omega-\omega_0/(1-M_0\cos\theta))$$
$$\times \left[\frac{S\{\{(\omega_0/c)(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \omega_0\}/(1-M_0\cos\theta)\}}{1+(\rho/\rho_0)(1-M\cos\theta)^2\sin\theta(1-M_0\cos\theta)/\psi(\theta)}\right]^2, \tag{60}$$

which shows that all sound is transmitted at the frequency of the source with the Doppler shift associated with its motion and for which functions such as (25) are evaluated for $\mathbf{g} = \mathbf{G}$, e.g. $\gamma(\mathbf{G}) = (\omega_0/c)\psi(\theta)/(1-M_0\cos\theta)$, where

$$\psi(\theta) \equiv \{(c_0/c)^2(1-M\cos\theta)^2 - (\cos^2\theta+\sin^2\theta\cos^2\phi)\}^{\frac{1}{2}}$$

(Howe 1975).

The general formula for the acoustic power (59) includes the case (60) of a plane interface obtained by Howe (1975), who has also considered statistical effects (Howe 1976). For turbulent diffraction the latter are embodied in the characteristic function (57) and autocorrelation coefficient (55) and (56), which have been shown to be consistent respectively (figure 3) with the experiments of Schmidt & Tilmann (1970) and Ho & Kovasznay (1976*a*, *b*). Altogether this may be regarded as a preliminary verification of the ideas underlying the present theory, so that we may conclude by expressing the results in a form suitable for subsequent, more detailed, applications. The non-trivial integrations $d\Omega\, d\omega$ in (59) imply that the field observed in the ambient medium is generally neither monochromatic nor omnidirectional, suggesting the definition of the spectral directivity I as the acoustic power per unit solid angle and unit frequency band:

$$dW = I(\theta,\phi,\omega)\, d\Omega\, d\omega,$$

$$J(\theta,\phi) = \int_{-\infty}^{+\infty} I(\theta,\phi,\omega)\, d\omega, \quad H(\omega) = \int_{\mathscr{U}} I(\theta,\phi,\omega)\, d\Omega, \tag{61a–c}$$

where $J(\theta,\phi)$ and $H(\omega)$ denote the total directivity and total spectrum respectively, which are obtained by integrating the spectral directivity over all frequencies or over the unit sphere $\mathscr{U}$. However, the spectrum $I(\theta_0, \phi_0; \omega)$ is generally not the same in each radiation direction (θ_0, ϕ_0), and similarly, the directivity $I(\theta, \phi; \omega_0)$ is different at distinct frequencies ω_0, so that a detailed specification of the sound field requires the spectral directivity.

Comparison with (59) shows that the spectral directivity involves three basic functions:

$$\Theta(\theta,\phi) = \frac{1/\rho_0 c_0^3}{256\pi^5}\frac{\sin^2\theta\cos^2\phi}{1-M_0\cos\theta}, \quad \Psi(\mathbf{g},\omega) = \omega^2\left[\frac{A(\mathbf{g})\,S(\mathbf{g})}{\gamma(\mathbf{g})}\right]^2, \tag{62a, b}$$

$$\Phi(\mathbf{g},\omega,\mathbf{G};\mathbf{z},\tau) = (\mathbf{g}-\mathbf{G}).\mathbf{z}+\{\omega-\omega_0-\mathbf{g}.\mathbf{U}+\alpha\mathbf{V}.(\mathbf{g}-\mathbf{G})\}\tau. \tag{62c}$$

These may be interpreted as follows: (*a*) the observation function $\Theta(\theta, \phi)$ depends on the properties of the ambient medium, on the motion of the source and on the direction of observation; (*b*) the amplitude function for the energy $\Psi(\mathbf{g}, \omega)$ depends on the field emitted by the source (S for each wave component $\mathbf{g}$), modified by propagation in the jet (25*b*) and transmission to the ambient medium (26*a*), and also involves the frequency of reception ω; (*c*) the phase function $\Phi(\mathbf{g}, \omega, \mathbf{G}; \mathbf{z}, \tau)$ depends on all the preceding quantities as well as on the spatial separation and time delay $(\mathbf{z}, \tau)$ and is associated with the difference between the horizontal wave vectors of emission and reception and a phase shift, the latter being equal to the difference between the frequencies of reception and emission, with one Doppler shift $\mathbf{g}\,.\,\mathbf{U}$ to account for source motion and another $\alpha V\,.\,(\mathbf{g}-\mathbf{G})$ to account for the mean convection of the shear layer. The characteristic function (57*a*) determines the degree of attenuation (57*b*) produced by interfacial irregularities and distributed turbulence, and also the interference between wave components within a correlation scale [see (55) and (56)].

The spectral directivity of sound transmitted through a single irregular and turbulent shear layer involves these four functions, with integrations over all propagating (incident and transmitted) wave components d^2g (γ, Γ real) and over the mean plane of the shear layer for all time $d^2z\,d\tau$:

$$I(\theta, \phi, \omega) = \Theta(\theta, \phi) \int^{\mathrm{Re}\gamma,\, \Gamma} \Psi(\mathbf{g}, \omega) \exp\{i\Phi(\mathbf{g}, \omega, \mathbf{G}; \mathbf{z}, \tau)\}\, C(\mathbf{z}, \tau)\, d^2g\, d^2z\, d\tau. \quad (63a)$$

The spectral directivity II for transmission through a double shear layer, which is obtained by an analysis similar to (58)–(62) after substitution of (35) into (42) as a starting formula, or more simply, by appropriate use of the functions (57*a*) and (62), is given by

$$II(\theta, \phi, \omega) = \{\Theta(\theta, \phi)/(2\pi)^3\} \int^{\mathrm{Re}\gamma,\, \gamma_+,\, \Gamma} \{\omega S(\mathbf{g})\, A_-(\mathbf{g})\, A(\mathbf{g}_+)/\gamma\}^2\, C(\mathbf{z}_+, \tau_+)\, C(\mathbf{z}_-, \tau_-)$$
$$\times \exp[i\{\Phi(\mathbf{G}, \omega, \mathbf{g}_+; \mathbf{z}_+, \tau_+) + \Phi(\mathbf{g}_+, \omega_+, \mathbf{g}; \mathbf{z}_-, \tau_-)\}]\, d^2g_-\, d^2z_-\, d\tau_-\, d\omega_+\, d^2g_+\, d^2z_+\, d\tau_+. \quad (63b)$$

Similarly, the energy radiated through any system of shear layers, whether single, double or multiple, and involving reflexions, transmissions or both, may be readily written down. For example, the energy associated with transmission through a double shear layer involving intermediate internal reflexions may be added to (63*b*), the nth-order correction involving $11+12n$ integrals. For a system of plane interfaces devoid of turbulence $C = 1$ and all integrations may be performed trivially and reduce essentially to multiplication by the scattering factors in (4*a*, *b*), which describe the redistribution of energy, the spectrum being unchanged throughout the refraction process. For a turbulent and/or irregular shear layer $C = C(\mathbf{z}, \tau)$ and refraction integrals of the form (63*a*, *b*) imply that not only is the energy redirected spatially, allowing radiation into regions inaccessible for scattering by a plane interface (because the conditions $\mathrm{Re}\,\gamma$, Γ apply only locally), but also the incident spectrum, even if monochromatic, will generally be broadened into a band of frequencies. Also, since $C < 1$, which means that some of the acoustic energy is absorbed, deflected or backscattered by the corresponding components of the turbulence spectrum (4.2), it can generally be predicted that a system of turbulent and/or irregular shear layers will transmit *less* sound than would the corresponding set of plane interfaces devoid of turbulence when placed between the same media. This property applies to the energy transmitted over all frequencies in each direction in which the plane system can radiate, i.e. the directivity is reduced by the presence of interfacial irregularities or distributed turbulence.

I record here the benefit I have derived from discussions with my supervisor, Dr M. S. Howe, on the subject of scattering of waves. I owe to Professor Sir James Lighthill much of the encouragement for the present study. The work was performed while the author was on leave from the Instituto Superior Técnico, Lisbon, Portugal, and also also supported by a scholarship from the Portuguese Ministry of Education.

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